A Brief Introduction to the Atomic Structure of Puiseux Monoids

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What is a Puiseux Monoid?

Definition

A **Puiseux monoid** is an additive submonoid of $(\mathbb{Q}_{\geq 0}, +)$.

- Every Puiseux monoid M is generated by a non-empty subset $P \subseteq \mathbb{Q}_{\geq 0}$:

$$M = \langle P \rangle = \left\{ \sum_{i=1}^{n} c_i q_i \;\middle|\; n \in \mathbb{N}, c_i \in \mathbb{N}_0, q_i \in P \right\}$$

- Puiseux monoids are commutative, reduced (only 0 is invertible), and cancellative.
- They bridge additive combinatorics, commutative monoid theory, and semiring algebra.

Atomicity in Monoids

Let M be a reduced, commutative monoid.

Definition

An element $a \in M \setminus \{0\}$ is an **atom** (or irreducible) if $a = b + c$ implies $b = 0$ or $c = 0$. The set of all atoms of M is denoted by $\mathcal{A}(M)$.

Definition

M is **atomic** if every non-zero element can be written as a finite sum of atoms, i.e., $M = \langle \mathcal{A}(M) \rangle$.

Example (Atoms in Puiseux Monoids)

If $M = \left\langle \frac{1}{2^n} \mid n \in \mathbb{N}_0 \right\rangle$, then $\mathcal{A}(M) = \emptyset$, so M is non-atomic (it is an antimatter monoid).

Generating Sets and Atomicity

Theorem

Let M be a Puiseux monoid.

1. *M possesses a unique minimal generating set, denoted by G_M.*
2. *M is atomic if and only if $G_M = \mathcal{A}(M)$.*

Key Difference from General Monoids

Unlike general commutative monoids, for any Puiseux monoid M, the minimal generating set G_M exists and is unique. Thus, testing atomicity is equivalent to checking whether G_M consists entirely of irreducibles.

Conditioning on Generating Sets

Let $P \subset \mathbb{Q}_{>0}$ be the minimal generating set of $M = \langle P \rangle$.

Finitely Generated Puiseux Monoids

If $|P| < \infty$, then M is always atomic (isomorphic to a numerical monoid).

Increasingly Generated Monoids

If $P = \{p_n\}_{n=1}^{\infty}$ is an increasing sequence of rational numbers ($p_1 < p_2 < p_3 < \dots$), then M is atomic, and $\mathcal{A}(M) = P$.

Decreasing Sequences

Decreasing sequences of generators often fail atomicity or produce non-atomic/antimatter behavior!

Example 1: Atomic vs. Antimatter Behavior

Example (Increasingly Generated Puiseux Monoid)

Consider $M_1 = \left\langle \frac{n}{n+1} \;\middle|\; n \in \mathbb{N} \right\rangle = \langle \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots \rangle$.

- The sequence of generators $P = \{\frac{n}{n+1}\}_{n=1}^{\infty}$ is strictly increasing.
- Hence M_1 is **atomic**, and $\mathcal{A}(M_1) = P$.

Example (Antimatter Monoid)

Consider $M_2 = \langle \frac{1}{3^n} \mid n \in \mathbb{N}_0 \rangle = \langle 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots \rangle$.

- For any generator $\frac{1}{3^n}$, we can write $\frac{1}{3^n} = \frac{1}{3^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{3^{n+1}}$.
- M_2 contains no atoms ($\mathcal{A}(M_2) = \emptyset$), making it an **antimatter** monoid.

Factorization Properties

For $x \in M$, let $Z(x)$ be the set of factorizations of x into atoms.

Definition

- M is a **Bounded Factorization Monoid (BFM)** if for every $x \in M$, the lengths of all factorizations in $Z(x)$ are bounded.
- M is a **Finite Factorization Monoid (FFM)** if $Z(x)$ is finite for every $x \in M$.

Theorem

For Puiseux monoids, the following implications hold:

$$\textit{FFM} \implies \textit{BFM} \implies \textit{Atomic}$$

Characterizing FFMs and BFMs

Let $q \in \mathbb{Q}_{>0}$ with $q = a/b$ in lowest terms. Define $d(q) = b$.

Theorem (Characterization of BFMs and FFMs)

Let $M = \langle P \rangle$ be a Puiseux monoid.

- *M is a* **BFM** *if and only if 0 is not a limit point of P.*
- *M is an* **FFM** *if and only if for every $x \in M$, the set $\{p \in P \mid p \leq x \text{ and } d(p) \text{ divides } \mathrm{denom}(x)\}$ is finite.*

Example

If $P = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$, then 0 is a limit point of P, so M is not a BFM.

Example 2: BFMs and Limit Points of Generators

Example (A Non-BFM Atomic Monoid)

Let $P = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$ and $M = \langle P \rangle$.

- The set of generators P has 0 as a limit point ($\lim_{n\to\infty} \frac{1}{n} = 0$).
- Consider the element $1 \in M$. For any $n \in \mathbb{N}$, $1 = n \cdot \frac{1}{n}$ is a factorization into n atoms.
- Thus, factorization lengths of 1 are unbounded: M is **not a BFM**.

Example (A BFM Puiseux Monoid)

Let $P = \{1 + \frac{1}{n} \mid n \in \mathbb{N}\} = \{\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots\}$.

- The limit point of P is $1 > 0$, so $0 \notin \overline{P}$.
- $M = \langle P \rangle$ is a **BFM**.

Elasticity

Definition

For an atomic monoid M and $x \in M \setminus \{0\}$, the **elasticity** of x is:

$$\rho(x) = \frac{\max\{\text{length}(z) \mid z \in Z(x)\}}{\min\{\text{length}(z) \mid z \in Z(x)\}}$$

The elasticity of M is $\rho(M) = \sup\{\rho(x) \mid x \in M \setminus \{0\}\}$.

Theorem

There exist Puiseux monoids with arbitrary rational or infinite elasticity. Furthermore, full elasticity spectrum results can be constructed via controlled generator denominator growth.

Special Classes of Puiseux Monoids

Primary Puiseux Monoids

Puiseux monoids generated by fractions whose denominators are powers of a single prime p.

Grams' Monoid

A classical example of an atomic monoid that is not a BFM (and whose localizations exhibit non-trivial arithmetic behavior):

$$M = \left\langle \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$$

where p_n is the n-th odd prime.

Example 3: Non-Unique Factorizations in Grams' Monoid

Let $M = \left\langle q_n = \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$, where p_n is the n-th odd prime $(p_1 = 3, p_2 = 5, p_3 = 7, \dots)$.

Properties of Grams' Monoid

- M is **atomic** with minimal generating set $\mathcal{A}(M) = \{q_n \mid n \in \mathbb{N}\}$.
- 0 is a limit point of $\mathcal{A}(M)$, so M is **not a BFM**.

Example (Multiple Factorizations)

The element $x = \frac{1}{6} \in M$ can be represented as:

$$x = q_1 = \frac{1}{2 \cdot 3} \quad \text{(length 1)}$$

By expressing $1/6$ using generators with larger denominators $(q_2, q_3, \dots)$, one can construct factorizations of arbitrarily large

Open Questions & Further Directions

Related Open Problems

1. Given a field F, characterize when the monoid algebra $F[M]$ or semiring $S[M]$ inherits atomicity from certain relevant Puiseux monoids M.
   - Is $\mathbb{Q}[M]$ atomic when M is the Grams' monoid?
   - Is $\mathbb{Q}[M]$ atomic when $M := \{(2/3)^n : n \in \mathbb{N}_0\}$?
2. Given a field F, does the FF-property ascend from a Puiseux monoid M to its monoid algebra $F[M]$? Can we solve at least the special cases of $F = \mathbb{Q}$ and $F = \mathbb{F}_p$?.
3. Is it true that if a Puiseux monoid M is a BFM then every submonoid of M is atomic?

References & Conclusion

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The End

Thank You!

Questions / Comments